

# Critical end point in a thermo-magnetic nonlocal NJL model

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## Abstract

In this article we explore the critical end point in the  $T - \mu$  phase diagram of a thermomagnetic nonlocal Nambu–Jona-Lasinio model in the weak field limit. We work with the Gaussian regulator, and find that a crossover takes place at  $\mu, B = 0$ . The crossover turns to a first order phase transition as the chemical potential or the magnetic field increase. The critical end point of the phase diagram occurs at a higher temperature and lower chemical potential as the magnetic field increases. This result is in accordance to similar findings in other effective models. We also find there is a critical magnetic field, for which a first order phase transition takes place even at  $\mu = 0$ .

*Keywords:*

QCD, Critical End Point, nonlocal Nambu–Jona-Lasinio, Phase Diagram.

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## 1. Introduction

It is well known that the chiral phase transition of hadronic matter in QCD is a first order one at high enough values of the chemical potential. At low or vanishing  $\mu$ , it is unclear if the phase transition is either a second order one or a crossover. A critical end point (CEP) then separates both regimes, the existence of the CEP in QCD was suggested a few decades ago [1, 2, 3, 4]. The QCD phase diagram and its properties fall within the nonperturbative realm of QCD and are therefore studied through a number of different approaches. Particu-

lary, the position of the critical end point has been studied through lattice QCD [5, 6] and the use of effective models as the linear sigma model [7], the Nambu–Jona-Lasinio (NJL) model [8, 9, 10, 11] and its nonlocal variant (nNJL) [12, 13].

More recently, the study of the QCD phase diagram in the presence of a magnetic field has been addressed in many articles [14, 15, 16, 17, 18, 19, 20, 21, 7, 22, 23, 24, 25, 26]. The magnetic field has been shown to have an effect on both the order of the phase transition and the critical temperature and chemical potential at which it occurs [7]. The magnetic field then has an effect on the position of the critical end in the phase diagram. In this article we study the effect of the magnetic field on the critical end point of the phase diagram of a nNJL model.

The NJL model was originally proposed as model of interacting nucleons [27, 28] and later reinterpreted as a model of interacting quarks [29, 30]. Later, a nonlocal variant of the model, the nNJL model, was proposed. The nonlocal aspects of the nNJL model allow for an incorporation of quark confinement through the quasiparticle interpretation of the singularities of the propagator [31, 32, 33].

The paper is organized as follows. In Sec. 2, the model is presented and the uniform magnetic field is introduced. In Sec. 3 our results are presented and in Sec. 4 we discuss our conclusions and final remarks.

## 2. Thermo-magnetic nNJL Model.

The nNJL model is described through the Euclidean Lagrangian

$$\mathcal{L}_E = \left[ \bar{\psi}(x)(-i\not{D} + m)\psi(x) - \frac{G}{2}j_a(x)j_a(x) \right], \quad (1)$$

with  $\psi(x)$  being the quark field. The nonlocal aspects of the model are incorporated through the nonlocal currents  $j_a(x)$

$$j_a(x) = \int d^4y d^4z r(y-x)r(z-x)\bar{\psi}(x)\Gamma_a\psi(z), \quad (2)$$

where  $\Gamma_a = (1, i\gamma^5\vec{\tau})$  and  $r(x)$  is the so-called regulator of the model. It is usual to bosonize the model through the incorporation of a scalar ( $\sigma$ ) and a pseudoscalar ( $\vec{\pi}$ ) field. Then, in the mean field approximation,

$$\sigma = \bar{\sigma} + \delta\sigma \quad (3)$$

$$\vec{\pi} = \delta\vec{\pi}, \quad (4)$$

where  $\bar{\sigma}$  is the vacuum expectation value of the scalar field, serving as an order parameter for the chiral phase transition. The vacuum expectation value of the pseudoscalar field was taken to be null because of isospin symmetry. Quark fields can then be integrated out of the model [34, 35] and the mean field effective action can be obtained

$$\Gamma^{MF} = V_4 \left[ \frac{\bar{\sigma}^2}{2G} - 2N_c \int \frac{d^4q_E}{(2\pi)^4} \text{tr} \ln S_E^{-1}(q_E) \right], \quad (5)$$

with  $S_E(q_E)$  being the Euclidean effective propagator

$$S_E = \frac{-\not{q}_E + \Sigma(q_E^2)}{q_E^2 + \Sigma^2(q_E^2)}. \quad (6)$$

Here,  $\Sigma(q_E^2)$  is the constituent quark mass

$$\Sigma(q_E^2) = m + \bar{\sigma}r^2(q_E^2). \quad (7)$$

Finite temperature ( $T$ ) and chemical potential ( $\mu$ ) effects can be incorporated through the ITF or Matsubara formalism. To do so, one can make the following substitutions

$$V_4 \rightarrow V/T \quad (8)$$

$$q_4 \rightarrow -q_n \quad (9)$$

$$\int \frac{dq_4}{2\pi} \rightarrow T \sum_n, \quad (10)$$

where  $q_n$  includes the Matsubara frequencies

$$q_n \equiv (2n + 1)\pi T + i\mu. \quad (11)$$

With this, the propagator in Eq. (6) will now look like

$$S_E(q_n, \mathbf{q}, T) = \frac{\gamma^4 q_n - \boldsymbol{\gamma} \cdot \mathbf{q} + \Sigma(q_n, \mathbf{q})}{q_n^2 + \mathbf{q}^2 + \Sigma^2(q_n, \mathbf{q})}. \quad (12)$$

It is worth noting that the propagator in Eq. (12) has no singularities. Since there are no poles at some  $p^2$ , the definition of an effective mass for the particle with such propagator is not clear and therefore the quasiparticle interpretation cannot be made.

The  $\sigma$  field will evolve with temperature. This evolution can be computed through the grand canonical thermodynamical potential in the mean field approximation  $\Omega_{MF}(\bar{\sigma}, T, \mu) = (T/V)\Gamma_{MF}(\bar{\sigma}, T, \mu)$  [36]. Then the value of  $\bar{\sigma}$  must be at the minimum of the potential where  $\partial\Omega_{MF}/\partial\bar{\sigma} = 0$ , which means

$$\frac{\bar{\sigma}}{G} = 2N_c T \sum_n \int \frac{d^3q}{(2\pi)^3} r^2(q_E^2) \text{tr} S_E(q_E) \Big|_{q_4 = -q_n}. \quad (13)$$

From this equation one can get the temperature evolution of  $\bar{\sigma}$ . So far, all of the computations have been made in ITF. Similar derivations are readily available in the literature (see for example [34, 37]).

We are interested in studying the model coupled to a homogeneous magnetic field. The derivative in the Lagrangian (1) is replaced by a covariant derivative

$$D_\mu = \partial_\mu + ie_f A_\mu. \quad (14)$$

where  $A^\mu$  is the vector potential corresponding to an homogeneous external magnetic field  $\mathbf{B} = |\mathbf{B}|\hat{z}$  and  $e_f$  is the electric charge of the quark fields (i.e.  $e_u = 2e/3$  and  $e_d = -e/3$ ). In the symmetric gauge,

$$A^\mu = \frac{B}{2}(0, -y, x, 0), \quad (15)$$

The Schwinger proper time representation for the propagator is given by [38]

$$S(q) = -i \int_0^\infty ds \frac{e^{-is(M^2 - q_\parallel^2 + q_\perp^2 \frac{\tan(eBs)}{eBs})}}{\cos(eBs)} \times \left[ (\cos(eBs) + \gamma_1 \gamma_2 \sin(eBs)) (M + \not{q}_\parallel) - \frac{\not{q}_\perp}{\cos(eBs)} \right], \quad (16)$$

with  $q_\parallel^2 = q_0^2 + q_3^2$ ,  $q_\perp^2 = q_1^2 + q_2^2$  and where  $e$  is the charge of the particle being  $B$  the magnetic field.

For simplicity, we will consider the weak magnetic field case. The fermionic propagator in this region can be written as [39]

$$S(q) = i \frac{(\not{q} + \Sigma(-q^2))}{q^2 - \Sigma^2(-q^2)} - \frac{\gamma_1 \gamma_2 (eB) (\not{q}_\parallel) + \Sigma(-q^2)}{(q^2 - \Sigma^2(-q^2))^2} - \frac{2i(eB)^2 q_\perp^2}{(q^2 - \Sigma^2(-q^2))^4} \times \left[ (\Sigma(-q^2) + \not{q}_\parallel) + \frac{\not{q}_\perp (\Sigma^2(-q^2) - q_\parallel^2)}{q_\perp^2} \right]. \quad (17)$$

### 3. Results

Throughout this work, we will use the Gaussian regulator for the nNJL model, i.e.

$$r^2(-q^2) = e^{-q^2/\Lambda^2}. \quad (18)$$

For the parameters of the model, we take [40]  $m = 10.5$  MeV,  $\Lambda = 627$  MeV and  $G = 5 \times 10^{-5}$  MeV<sup>2</sup>. With this set of parameters we have  $\bar{\sigma}_0 = 339$  MeV.

We solved the gap equation for different values of the magnetic field and the chemical potential, obtaining the behavior of  $\bar{\sigma}(T)$  for each pair of  $(eB, \mu)$  values. This allows to determine the critical temperature for chiral phase transition, as well as the nature of the phase transition, namely if it is a first or second order phase transition, or rather a crossover.

Figure 1 shows the  $T - \mu$  phase diagram for different values of the magnetic field. In all cases, we have a crossover that, at high enough chemical potential,

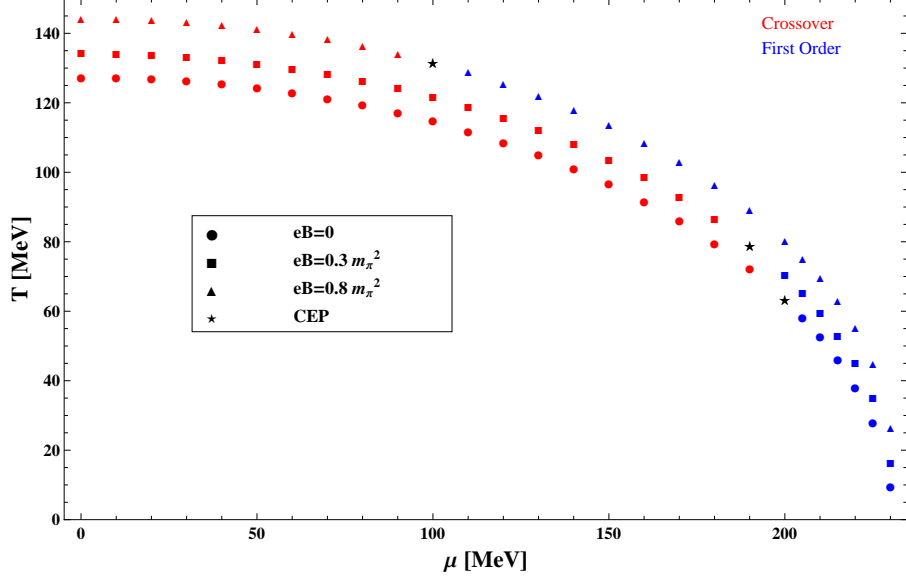


Figure 1:  $T - \mu$  phase diagram of the model, for different values of the magnetic field. Values of  $eB$  are indicated in the figure. Red points signal a crossover while blue ones signal a first order phase transition. The stars show the position of the critical end point.

turns to a first order phase transition, from the chirally broken, to the chirally restored phase. We can also see that the critical end point (CEP), namely, the point that separates the two kinds of transition, moves to the left of the phase diagram as the magnetic field increases. Similar results have been found in [7]. This means that for higher magnetic fields, the critical temperature increases while the critical chemical potential decreases.

Figure 2 shows the behavior of the chemical potential for the CEP as a function of the magnetic field. As can be seen from the figure, the chemical potential decreases as the magnetic field increases. Furthermore, at  $eB = 1.1 \text{ MeV}^2$ , the chemical potential for the critical endpoint is null, meaning that there is no longer a crossover in the phase diagram, but rather a first order phase transition at every critical temperature, therefore we can no longer define a CEP.

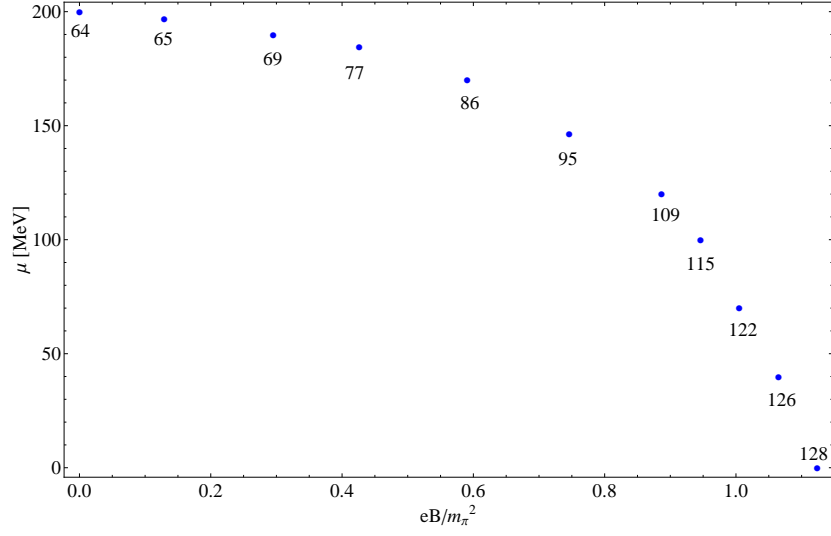


Figure 2: Behavior of the chemical potential of the critical end point as a function of the magnetic field. The number below data points indicate the value of the temperature of the CEP for each case.

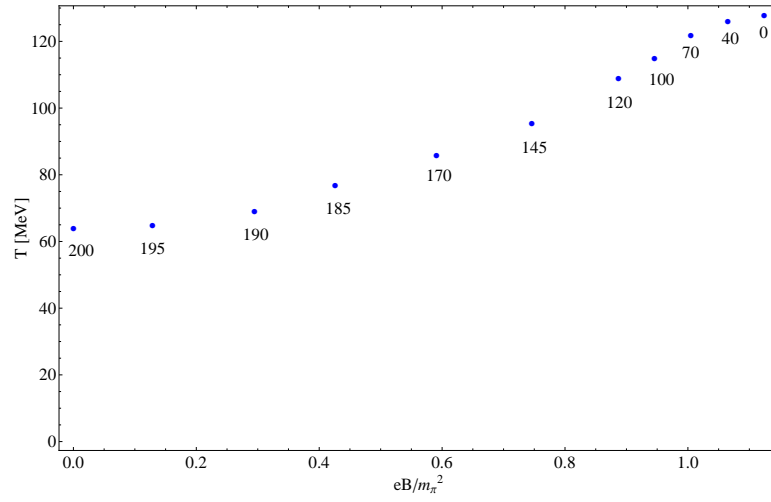


Figure 3: Behavior of the temperature of the critical end point as a function of the magnetic field. The number below data points indicate the value of the chemical potential of the CEP for each case.

Figure 3 shows the behavior of the temperature of the CEP as a function of the magnetic field. As can be seen from the figure, the temperature of the CEP increases as a function of the magnetic field. However, at  $T = 128$  MeV, the CEP can no longer be defined, as there is no longer a crossover in the model.

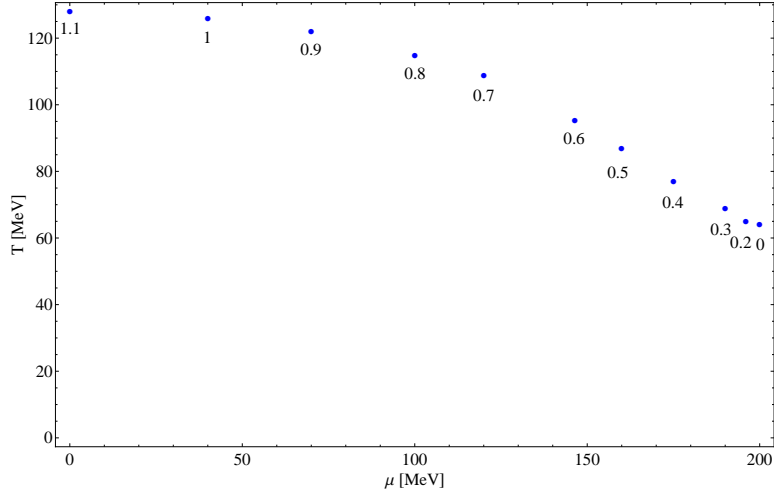


Figure 4: Behavior of the temperature of the CEP as a function of the chemical potential of the CEP. The number below data points indicate the value of the magnetic field for each case.

Figure 4 shows the behavior of the temperature of the CEP as a function of the chemical potential of the CEP. As the chemical potential increases, the temperature decreases.

#### 4. Conclusions

In this article, we obtained the  $T - \mu$  phase diagram for thermo-magnetic nNJL model. We find that there is a chiral phase transition that can either be a crossover or a first order phase transition at  $B = 0$ , dependind on the value of the chemical potential. One can then define a critical end point as the set of  $(T, \mu, eB)$  values that separate the crossover from the first order phase transition. We find that as the magnetic field increases the temperature of the



CEP also increases, while the chemical potential of the CEP decreases. If the magnetic field is high enough, the chemical potential of the CEP will vanish, meaning that we will no longer have a crossover in our model, but rather a phase transition for any temperature. At this point is no longer possible to define a CEP.

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